JOURNAL OF APPROXIMATION THEORY 68, 223–246 (1992)

Antisymmetric Tensor Products of Absolutely p-Summing Operators

STEFAN GEISS

Mathematische Fakultät der Friedrich Schiller Universität Jena, UHH 17.OG, Jena, D 0-6900, Germany

Communicated by Paul L. Butzer

Received October 15, 1990

We consider antisymmetric tensor products of absolutely p-summing operators. In connection with this second moments of determinants of random matrices appear. These second moments are closely related to approximation properties of the absolutely 2-summing operators and can be used to characterize some classes of infinite-dimensional Banach spaces. Finite-dimensional results are also obtained by this approach.

The starting point of the present paper is the result of Holub [4], which says that the injective tensor product of two absolutely p-summing operators S and T is again absolutely p-summing whereas $\pi_p(S \otimes_\varepsilon T) \leqslant \pi_p(S)\,\pi_p(T)$.

We consider *antisymmetric* injective tensor products of an absolutely p-summing operator and prove norm estimates that are better than those appearing in the general case covered by Holub; see Theorems 1.1 and 1.2. In connection with this, second moments of determinants of random matrices appear and suggest a modified definition of the Grothendieck numbers.

For a linear and continuous operator S between Banach spaces X and Y these modified Grothendieck numbers are defined by

$$\Gamma_n(S;\mu) := \sup \left\{ \int_{B_{Y'}} \cdots \int_{B_{Y'}} |\det(\langle Sx_i, b_j \rangle)|^2 \, d\mu(b_1) \cdots d\mu(b_n) \right\}^{1/2n},$$

where the supremum is taken over all normalized elements $x_1, ..., x_n \in X$ and where μ is a normalized regular Borel measure on the unit ball of Y' equipped with the $\sigma(Y', Y)$-topology.

Some basic properties and examples of these quantities can be found in Section 2. In Section 3 we see that the modified Grothendieck numbers of S are closely related to the approximation numbers of the composition

0021-9045/92 $3.00

JS, where J: $Y \to L_2(B_{Y'}; \mu)$ is the canonical embedding. Using this fact we characterize some classes of Banach spaces in Theorems 3.2–3.4. In the last section we exploit finite-dimensional estimates of the modified Grothendieck numbers to reprove a result of Pełczynski and Szarek [9] concerning cubical volume ratios of convex and symmetric bodies in $\mathbb{R}^n$. With the same method we sharpen the relation between volume ratios of convex and symmetric bodies using ellipsoids of minimal and maximal volume and their analytical counterpart, the Grothendieck numbers; see Corollaries 4.3 and 4.4.

Preliminaries

If nothing is stated to the contrary, all Banach spaces are assumed to be real or complex. The closed unit ball of a Banach space X is denoted by B_X, the dual of X by X'. I_X is the identity-operator. The notations of special sequence and function spaces are adopted from [6]. The space of all linear and continuous operators from a Banach space X into a Banach space Y is denoted by $\mathscr{L}(X, Y)$ and equipped with the norm

$$\|S\| := \sup\{\|Sx\| : x \in B_X\}.$$

If K is a compact Hausdorff space then $W(K)$ denotes the set of all normalized regular Borel measures on K.

Let $1 \leqslant p < \infty$. An operator $S \in \mathscr{L}(X, Y)$ is *absolutely p-summing* if there exist $\mu \in W(B_{X'})$ ($B_{X'}$ is equipped with the $\sigma(X', X)$-topology) and a constant $c \geqslant 0$ such that

$$\|Sx\| \leqslant c \left(\int_{B_{X'}} |\langle x, a \rangle|^p \, d\mu(a) \right)^{1/p} \qquad \text{for all} \quad x \in X. \tag{$*$}$$

The space $\Pi_p(X, Y)$ of the absolutely p-summing operators from X into Y is endowed with the norm $\pi_p(S) := \inf c$, where the infimum is taken over all $c \geqslant 0$ such that $(*)$ holds for some measure μ.

Since the infimum is attained (see [10, (17.3.2)]) we may say that S *is dominated by* μ if $(*)$ is satisfied with $c = \pi_p(S)$.

An operator $S \in \mathscr{L}(X, Y)$ belongs to the class Γ_2^* if there exists a factorization $S = S_2 S_1$ through a Hilbert space H with $S_1 \in \Pi_2(X, H)$ and $S_2' \in \Pi_2(Y', H')$. As norm we set

$$\gamma_2^*(S) := \inf\{\pi_2(S_1)\, \pi_2(S_2')\},$$

where the infimum is taken over all possible representations. According to [10, (17.4.3)] the infimum is attained again. So we analogously say that S

is dominated by $\mu \in W(B_{X'})$ and $\nu \in W(B_{Y''})$ if $S = S_2 S_1$ and $\gamma_2^*(S) = \pi_2(S_1)\,\pi_2(S_2')$, where S_1 and S_2 are dominated by μ and ν, respectively.

Finally, we call an operator $S \in \mathscr{L}(X, Y)$ *nuclear* if there exist sequences $\{a_n\}_{n=1}^{\infty} \subset X'$ and $\{y_n\}_{n=1}^{\infty} \subset Y$ such that

$$Sx = \sum_{n=1}^{\infty} \langle x, a_n \rangle\, y_n \qquad \text{for all} \quad x \in X$$

and $\sum_{n=1}^{\infty} \|a_n\|\, \|y_n\| < \infty$. The set of all nuclear operators S from X into Y is denoted by $N(X, Y)$ and is a Banach space under

$$\nu(S) := \inf \left\{ \sum_{n=1}^{\infty} \|a_n\|\, \|y_n\| \right\}.$$

More information about the above operator classes can be found in [10] or [11].

1. Antisymmetric Tensor Products of Absolutely p-Summing Operators

Let X be a Banach space and S_n be the group of all permutations of the set $\{1, ..., n\}$. For X we define the nth *outer product* as

$$\Lambda^n X := \operatorname{span} \left\{ x_1 \wedge \cdots \wedge x_n := \sum_{S_n} \operatorname{sgn} \sigma x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \right\} \subseteq \otimes^n X$$

and denote the closure in the injective tensor product $\otimes_\varepsilon^n X$ by $\Lambda_\varepsilon^n X$. In the special situation in which $X = H$ is a Hilbert space we use

$$(s, t) := \sum_{i,j} \det((f_{ki}, g_{lj}))_{k,l=1}^n,$$

$$\left(s = \sum_i f_{1i} \wedge \cdots \wedge f_{ni} \in \Lambda^n H,\ t = \sum_j g_{1j} \wedge \cdots \wedge g_{nj} \in \Lambda^n H \right)$$

as scalar product on $\Lambda^n H$ and form the corresponding closure $\Lambda_2^n H$. The usual norm of an element $s \in \Lambda_\varepsilon^n X$ is denoted by $\varepsilon(s)$ (see below). In $\Lambda_2^n H$ we take $\tau(s) := (s, s)^{1/2}$.

The elements of $\Lambda_\varepsilon^n X$ can be naturally considered as antisymmetric functionals on $X' \times \cdots \times X'$ in the following way. To each $s \in \Lambda^n X$ with $s = \sum_i x_{1i} \wedge \cdots \wedge x_{ni}$ we assign a continuous functional

$$\tilde{s} \in \mathbb{L}(X', ..., X') := \{t\colon X' \times \cdots \times X' \to \mathbb{R}, \mathbb{C}\colon n\text{-linear and continuous}\}$$

by

$$\tilde{s}(a_1, ..., a_n) := \sum_i \det(\langle x_{ki}, a_l \rangle)_{k,l=1}^n.$$

$\tilde{s}$ does not depend on the special representation of s and

$$\|\tilde{s}\| = \sup\{|\tilde{s}(a_1, ..., a_n)| : a_i \in B_{X'}\} = \varepsilon(s).$$

Hence $\Lambda_\varepsilon^n X$ is an isometric subspace of $\mathbb{L}(X', ..., X')$ and

$$\|s\|_{L_p(\mu^n)} := \left(\int_{B_{X'}} \cdots \int_{B_{X'}} |s(a_1, ..., a_n)|^p \, d\mu(a_1) \cdots d\mu(a_n) \right)^{1/p}$$

is justified for $\mu \in W(B_{X'})$, $s \in \Lambda_\varepsilon^n X$, and $1 \leqslant p < \infty$.

Furthermore, assuming K to be a compact Hausdorff space we recall $\bigotimes_\varepsilon^n C(K) = C(K \times \cdots \times K)$ and deduce

$$\Lambda_\varepsilon^n C(K) = C^a(K \times \cdots \times K),$$

where $C^a(K \times \cdots \times K)$ is the subspace of $C(K \times \cdots \times K)$ consisting of all antisymmetric and continuous functions on $K \times \cdots \times K$.

Finally, we introduce the *outer* ε*-product* $\Lambda_\varepsilon^n S$: $\Lambda_\varepsilon^n X \to \Lambda_\varepsilon^n Y$ of an operator $S \in \mathscr{L}(X, Y)$ by $(\Lambda_\varepsilon^n S)(x_1 \wedge \cdots \wedge x_n) := Sx_1 \wedge \cdots \wedge Sx_n$. The *outer* 2*-product* $\Lambda_2^n S$ of an operator S acting between Hilbert spaces is defined analogously.

Now we can formulate the main results.

THEOREM 1.1. *Let* $2 \leqslant p < \infty$ *and* $S \in \Pi_p(X, Y)$ *be dominated by the measure* μ. *Then*

$$\varepsilon((\Lambda_\varepsilon^n S)s) \leqslant n!^{-1/p} \, \pi_p(S)^n \, \|s\|_{L_p(\mu^n)} \qquad \text{for all} \quad s \in \Lambda_\varepsilon^n X.$$

Consequently, $\pi_p(\Lambda_\varepsilon^n S) \leqslant n!^{-1/p} \, \pi_p(S)^n$.

THEOREM 1.2. *Let* H *be a Hilbert space and let* $S \in \Pi_2(X, H)$ *be dominated by the measure* μ. *Then* $\Lambda^n S$: $\Lambda_\varepsilon^n X \to \Lambda_2^n H$ (*induced in the canonical way*) *is absolutely* 2*-summing with*

$$\tau((\Lambda^n S)s) \leqslant n!^{-1/2} \, \pi_2(S)^n \, \|s\|_{L_2(\mu^n)} \qquad \text{for all} \quad s \in \Lambda_\varepsilon^n X.$$

Consequently, $\pi_2(\Lambda^n S) \leqslant n!^{-1/2} \, \pi_2(S)^n$.

To prove the above results we start with a formula which is of

Cauchy–Binet type and describe $\Lambda_2^n L_2(\Omega, \mu)$ in the case when $(\Omega, \mathscr{F}, \mu)$ is a σ-finite measure space. For this purpose the linear map

$$\Psi\colon \Lambda^n L_2(\Omega, \mu) \to L_2(\Omega \times \cdots \times \Omega, \mu \times \cdots \times \mu)$$

is defined on the representatives by

$$\Psi(f_1 \wedge \cdots \wedge f_n) := ((\omega_1, ..., \omega_n) \to \det(f_i(\omega_j))_{i,j=1}^n).$$

Furthermore, for $1 \leqslant p < \infty$ we denote by $L_p^a(\Omega \times \cdots \times \Omega, \mu \times \cdots \times \mu)$ the closed subspace of $L_p(\Omega \times \cdots \times \Omega, \mu \times \cdots \times \mu)$ defined by

$$\{f \in L_p\colon \text{there exists an antisymmetric } f' \in f \text{ defined everywhere}\}.$$

LEMMA 1.3. *Let $(\Omega, \mathscr{F}, \mu)$ be a σ-finite measure space. Then*

$$\Lambda_2^n L_2(\Omega, \mu) = L_2^a(\Omega \times \cdots \times \Omega, \mu \times \cdots \times \mu),$$

where for all s, $t \in \Lambda^n L_2$

$$(s, t)_{\Lambda_2^n L_2} = n!^{-1} \int_\Omega \cdots \int_\Omega \Psi(s)(\omega_i)\, \overline{\Psi(t)(\omega_i)}\, d\mu(\omega_1) \cdots d\mu(\omega_n).$$

Proof. Let s, $t \in \Lambda^n L_2(\Omega, \mu)$ be given by $s = \sum_i f_{1i} \wedge \cdots \wedge f_{ni}$ and $t = \sum_j g_{1j} \wedge \cdots \wedge g_{nj}$. Then

$$\begin{aligned}
(s, t) &= \sum_{i,j} \det(f_{ki}, g_{lj})_{k,l=1}^n \\
&= \sum_{i,j} \det\left(\int_\Omega f_{ki}\, \overline{g_{lj}}\, d\mu\right)_{k,l} \\
&= \sum_{i,j} \sum_{S_n} \operatorname{sgn} \sigma \int \cdots \int f_{1i}(\omega_1) \cdots f_{ni}(\omega_n) \\
&\qquad \times \overline{g_{\sigma(1)j}(\omega_1) \cdots g_{\sigma(n)j}(\omega_n)}\, d\mu(\omega_1) \cdots d\mu(\omega_n) \\
&= \sum_{i,j} \int \cdots \int f_{1i}(\omega_1) \cdots f_{ni}(\omega_n) \\
&\qquad \times \overline{\det(g_{kj}(\omega_l))_{k,l}}\, d\mu(\omega_1) \cdots d\mu(\omega_n) \\
&= \sum_{i,j} n!^{-1} \int \cdots \int \det(f_{ki}(\omega_l))_{k,l} \\
&\qquad \times \overline{\det(g_{kj}(\omega_l))_{k,l}}\, d\mu(\omega_1) \cdots d\mu(\omega_n) \\
&= n!^{-1} \int \cdots \int \Psi s\, \overline{\Psi t}\, d\mu \cdots d\mu.
\end{aligned}$$

Hence Ψ: $\Lambda^n L_2(\Omega) \to L_2^a \subseteq L_2(\Omega \times \cdots \times \Omega)$ is an isometric embedding (with the factor $n!^{-1/2}$). To show that the extension $\tilde{\Psi}$: $\Lambda_2^n L_2 \to L_2^a$ is a surjection we approximate an element $f \in L_2^a$ by step-functions f_k in the L_2-norm. It is clear that we can assume

$$f_k = \sum_i \lambda_i \chi_{A_{1i}^k \times \cdots \times A_{ni}^k} \qquad (\lambda_i \in \mathbb{R}, \mathbb{C}, A_{li}^k \in \mathscr{F}).$$

Considering the operator alt: $L_2(\Omega \times \cdots \times \Omega) \to L_2(\Omega \times \cdots \times \Omega)$ defined on the representatives by $\operatorname{alt}(f) = n!^{-1} \sum_{S_n} \operatorname{sgn} \sigma f_\sigma$ $(f_\sigma(\omega_1, ..., \omega_n) := f(\omega_{\sigma(1)}, ..., \omega_{\sigma(n)}))$ we obtain $\|\operatorname{alt}\| \leqslant 1$ and

$$\operatorname{alt}(f_k) \xrightarrow[k]{} \operatorname{alt}(f) = f$$

in the L_2-norm. Since $\operatorname{alt}(f_k) \in \Psi(\Lambda^n L_2)$ we have $f \in \tilde{\Psi}(\Lambda_2^n L_2)$. ∎

LEMMA 1.4. *Let $2 \leqslant p < \infty$, K be a compact Hausdorff space and μ a regular measure. Then the map Φ: $C^a(K \times \cdots \times K) \to \Lambda_\varepsilon^n C(K)$ with*

$$\Phi((\omega_1, ..., \omega_n) \to \det(f_i(\omega_j))_{i,j=1}^n) = f_1 \wedge \cdots \wedge f_n$$

can be uniquely extended to a linear and continuous operator

$$\tilde{\Phi}: L_p^a(K \times \cdots \times K, \mu \times \cdots \times \mu) \to \Lambda_\varepsilon^n L_p(K, \mu).$$

Moreover $\|\tilde{\Phi}\| \leqslant n!^{-1/p}$.

Proof. First we mention that the inclusions

$$\Lambda^n C(K) \subseteq C^a(K \times \cdots \times K) \subseteq L_p^a(K \times \cdots \times K, \mu \cdots \times \mu)$$

are dense with respect to the L_p-norm. Let $1 = 1/p + 1/q$. Considering $s = \sum_i f_{1i} \wedge \cdots \wedge f_{ni} \in \Lambda^n C(K)$ we obtain

$$\begin{aligned}
\|\Phi s\|_{\Lambda_\varepsilon^n L_p} &= \sup \left\{ \left| \sum_i \det(\langle f_{ki}, g_l \rangle)_{k,l} \right| : \|g_l\|_q \leqslant 1 \right\} \\
&= \sup_{g_l} \left\{ n!^{-1} \left| \int \cdots \int s(\omega_1, ..., \omega_n) \det(g_l(\omega_k)) \, d\mu(\omega_1) \cdots d\mu(\omega_n) \right| \right\} \\
&\leqslant \{ n!^{-1} \|s\|_p \} \sup_{g_l} \left\{ \int \cdots \int |\det(g_l(\omega_k))|^q \, d\mu(\omega_1) \cdots d\mu(\omega_n) \right\}^{1/q}
\end{aligned}$$

from Lemma 1.3.

It remains to estimate the second factor from above by $n!^{1/q}$. If $q=2$ is taken Lemma 1.3 implies

$$\left(\int\cdots\int |\det(g_l(\omega_k))|^2\, d\mu(\omega_1)\cdots d\mu(\omega_n)\right)^{1/2}$$
$$= n!^{1/2}\, |\det((g_i, g_j))|^{1/2} \leqslant n!^{1/2}.$$

On the other hand, taking $q=1$ we use

$$\int\cdots\int |\det(g_l(\omega_k))|\, d\mu(\omega_1)\cdots d\mu(\omega_n)$$
$$\leqslant \sum_{S_n} \int\cdots\int |g_1(\omega_{\sigma(1)})\cdots g_n(\omega_{\sigma(n)})|\, d\mu(\omega_1)\cdots d\mu(\omega_n) \leqslant n!.$$

To treat the remaining case $1<q<2$ we consider the operator

$$M_q\colon L_q(K)\times\cdots\times L_q(K)\to L_q(K\times\cdots\times K)$$

defined (on the representatives) by

$$M_q(g_1, ..., g_n) := ((\omega_1, ..., \omega_n)\to \det(g_l(\omega_k))_{k,l=1}^n).$$

Now, for $1/q=\theta+(1-\theta)/2$ complex interpolation yields

$$\|M_q\| \leqslant \|M_1\|^\theta\, \|M_2\|^{1-\theta} \leqslant n!^\theta\, n!^{(1-\theta)/2} = n!^{1/q}. \quad \blacksquare$$

Now we are in a position to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Since S is dominated by μ there exist subspaces $X_0 \subseteq C(K)$, $X_p \subseteq L_p(K,\mu)$ $(K := B_{X'})$ and an operator $B\in\mathscr{L}(X_p, Y)$ with $\|B\|\leqslant \pi_p(S)$ such that

$$\begin{array}{ccc} X & \xrightarrow{\;S\;} & Y \\ {\scriptstyle A}\downarrow & & \uparrow{\scriptstyle B} \\ X_0 & \xrightarrow{\;J_p\;} & X_p \\ {\scriptstyle J}\downarrow & & \cap \\ C(K) & \xrightarrow{\;J_p\;} & L_p(K,\mu) \end{array}$$

where A is defined by $Ax := \langle x, \ \rangle$, J is the embedding of X_0 into $C(K)$

and J_p is the restriction of the embedding $\tilde{J}_p$. The injectivity of the ε-product implies the diagram

$$\begin{array}{ccc}
\Lambda^n_\varepsilon X & \xrightarrow{\Lambda^n_\varepsilon S} & \Lambda^n_\varepsilon Y \\
\downarrow{\scriptstyle \Lambda^n_\varepsilon A} & & \uparrow{\scriptstyle \Lambda^n_\varepsilon B} \\
\Lambda^n_\varepsilon X_0 & \xrightarrow{\Lambda^n_\varepsilon J_p} & \Lambda^n_\varepsilon X_p \\
\downarrow{\scriptstyle \Lambda^n_\varepsilon J} & & \cap| \\
\Lambda^n_\varepsilon C(K) & \xrightarrow{\Lambda^n_\varepsilon \tilde{J}_p} & \Lambda^n_\varepsilon L_p(K, \mu) \\
\| & & \uparrow{\scriptstyle \tilde{\Phi}_n} \\
C^a(K\times \cdots \times K) & \xrightarrow{\tilde{J}^n_p} & L^a_p(K\times \cdots \times K, \mu\times \cdots \times \mu)
\end{array}$$

where $\tilde{J}^n_p$ is the canonical embedding of C^a into L^a_p and $\tilde{\Phi}_n$ is the map from Lemma 1.4. We see

$$\begin{aligned}
\pi_p(\Lambda^n_\varepsilon S) &\leqslant \pi_p(\Lambda^n_\varepsilon J_p)\, \|\Lambda^n_\varepsilon B\| \leqslant \pi_p(\Lambda^n_\varepsilon \tilde{J}_p)\, \|B\|^n \\
&\leqslant \|\tilde{\Phi}_n\|\, \pi_p(\tilde{J}^n_p)\, \|B\|^n \leqslant n!^{-1/p}\, \pi_p(S)^n.
\end{aligned}$$

Furthermore, let $s \in \Lambda^n_\varepsilon X$. Then

$$\begin{aligned}
\varepsilon((\Lambda^n_\varepsilon S)s) &\leqslant \|\Lambda^n_\varepsilon B\|\, \varepsilon((\Lambda^n_\varepsilon J_p A)s) \leqslant \|\Lambda^n_\varepsilon B\|\, \varepsilon((\Lambda^n_\varepsilon \tilde{J}_p J A)s) \\
&\leqslant \|\Lambda^n_\varepsilon B\|\, \varepsilon(\tilde{\Phi}_n \tilde{J}^n_p(\Lambda^n_\varepsilon JA)s) \leqslant n!^{-1/p}\, \pi_p(S)^n\, \|s\|_{L_p(\mu^n)}. \quad \blacksquare
\end{aligned}$$

Proof of Theorem 1.2. Again setting $K = B_{X'}$ we can write the operator S as $S = BJA$, where $A \in \mathscr{L}(X, C(K))$ and $J \in \mathscr{L}(C(K), L_2(K, \mu))$ are the canonical embeddings and $B \in \mathscr{L}(L_2(K, \mu), H)$ satisfies $\|B\| \leqslant \pi_2(S)$. We obtain

$$\begin{array}{ccc}
\Lambda^n_\varepsilon X & \xrightarrow{\Lambda^n S} & \Lambda^n_2 H \\
\downarrow{\scriptstyle \Lambda^n_\varepsilon A} & & \uparrow{\scriptstyle \Lambda^n_2 B} \\
\Lambda^n_\varepsilon C(K) & & \Lambda^n_2 L_2(K, \mu) \\
\| & & \uparrow{\scriptstyle \Psi_n} \\
C^a(K\times \cdots \times K) & \xrightarrow{J^n_2} & L^a_2(K\times \cdots \times K, \mu\times \cdots \times \mu)
\end{array}$$

where $\tilde{\Psi}_n$ is taken from Lemma 1.3 with $\|\tilde{\Psi}_n\| = n!^{-1/2}$. As in the proof of Theorem 1.1 it follows that $\pi_2(\Lambda^n S) \leqslant n!^{-1/2}\, \pi_2(S)^n$ and

$$\tau((\Lambda^n S)s) \leqslant n!^{-1/2}\, \pi_2(S)^n\, \|s\|_{L_2(\mu^n)} \qquad \text{for all} \quad s \in \Lambda^n_\varepsilon X. \quad \blacksquare$$

To give a first corollary of Theorem 1.1 we define for $S\colon X \to Y \in \Pi_2$ and $T\colon Y \to X \in \Pi_2$ the determinant of $I + TS$ as

$$\det(I + TS) := 1 + \sum_{n=1}^{\infty} \operatorname{tr}(\Lambda^n_\varepsilon TS),$$

where tr is the unique continuous trace on the operator ideal Π_2^2 (see [11, (4.2.6)]).

Now we can improve [11, (4.7.17)] in the case $r = 1$.

Corollary 1.5. *Let $S \in \Pi_2(X, Y)$ and $T \in \Pi_2(Y, X)$. Then*

$$|\det(I + TS)| \leqslant \exp(\pi_2(T)\, \pi_2(S)).$$

Proof. Using [11, (4.2.6)] and Theorem 1.1 we see

$$\begin{aligned} |\det(I + TS)| &\leqslant 1 + \sum_{n=1}^{\infty} |\operatorname{tr}(\Lambda^n_\varepsilon TS)| \\ &\leqslant 1 + \sum_{n=1}^{\infty} \pi_2(\Lambda^n_\varepsilon T)\, \pi_2(\Lambda^n_\varepsilon S) \\ &\leqslant 1 + \sum_{n=1}^{\infty} n!^{-1}\, \pi_2(T)^n\, \pi_2(S)^n. \quad \blacksquare \end{aligned}$$

2. Modified Grothendieck Numbers

According to [2] the usual Grothendieck numbers of an operator $S \in \mathscr{L}(X, Y)$ are defined as

$$\begin{aligned} \Gamma_n(S) &:= \sup\{|\det(\langle Sx_i, b_j\rangle)_{i,j=1}^n|^{1/n} \colon x_i \in B_X,\, b_j \in B_{Y'}\} \\ &= \sup\{\varepsilon(Sx_1 \wedge \cdots \wedge Sx_n)^{1/n} \colon x_i \in B_X\}, \end{aligned}$$

whereas $\Gamma_n(X) := \Gamma_n(I_X)$.

Note that $\Gamma_n(X)$ measures the distance of the n-dimensional subspaces of

X to the Hilbert space by approximating the unit ball (of such a subspace) with the help of ellipsoids of maximal and minimal volume (see [3] and Corollary 4.4 of this paper).

Theorems 1.1 and 1.2 give rise to the following modification.

Let $S \in \mathcal{L}(X, Y)$ and $\mu \in W(B_{Y'})$. Then

$$\Gamma_n(S; \mu) := \sup \left\{ \int_{B_{Y'}} \cdots \int_{B_{Y'}} |\det(\langle Sx_i, b_j \rangle)_{i,j=1}^n|^2 \, d\mu(b_1) \cdots d\mu(b_n) \right\}^{1/2n},$$

where the supremum is taken over all $x_i \in B_X$. Again we use

$$\Gamma_n(X; \mu) := \Gamma_n(I_X; \mu).$$

In this section we present some basic properties and examples of these modified quantities "for fixed n," whereas in the next section we relate their asymptotic behaviour for "$n \to \infty$" to geometrical properties of the underlying Banach spaces.

For fixed n the usual and modified Grothendieck numbers satisfy

$$\left(\frac{n!}{n^n}\right)^{1/2n} \Gamma_n(S) \leqslant \sup\{\Gamma_n(S; \mu) : \mu \in W(B_{Y'})\} \leqslant \Gamma_n(S).$$

The right-hand inequality is clear. To see the left-hand one let $x_1, ..., x_n \in B_X$ and $b_1, ..., b_n \in B_{Y'}$ be arbitrary. Defining $\mu := 1/n \sum_{j=1}^n \delta_{b_j} \in W(B_{Y'})$, where δ_b is the Dirac measure at $b \in Y'$, we obtain

$$\begin{aligned} &\left(\frac{n!}{n^n}\right)^{1/2n} |\det(\langle Sx_i, b_j \rangle)|^{1/n} \\ &\quad = \left(\int_{B_{Y'}} \cdots \int_{B_{Y'}} |\det(\langle Sx_i, c_j \rangle)_{i,j=1}^n|^2 \, d\mu(c_1) \cdots d\mu(c_n) \right)^{1/2n} \\ &\quad \leqslant \Gamma_n(S; \mu). \end{aligned}$$

Taking the supremum over x_i and b_j we arrive at the desired result.

The following observations give more precise information about the interplay between the different Grothendieck numbers.

LEMMA 2.1. *Let $S \in \mathcal{L}(X, Y)$, $\mu \in W(B_{Y'})$, and $J\colon Y \to L_2(B_{Y'}; \mu)$ be the canonical embedding. Then*

$$\Gamma_n(JS) = n!^{-1/2n} \, \Gamma_n(S; \mu).$$

Proof. Applying Lemma 1.3 we obtain

$$\begin{aligned}
&\Gamma_n(JS)\\
&\quad=\sup\{|(JSx_1\wedge\cdots\wedge JSx_n, b_1\wedge\cdots\wedge b_n)_{\Lambda_2^n L_2}|^{1/n}: x_i\in B_X, b_j\in B_{L_2}\}\\
&\quad=\sup\{|(JSx_1\wedge\cdots\wedge JSx_n, JSx_1\wedge\cdots\wedge JSx_n)_{\Lambda_2^n L_2}|^{1/2n}: x_i\in B_X\}\\
&\quad=n!^{-1/2n}\sup_{x_i}\left\{\int_{B_{Y'}}\cdots\int_{B_{Y'}}|\det(\langle Sx_i, b_j\rangle)_{i,j=1}^n|^2\, d\mu(b_1)\cdots d\mu(b_n)\right\}^{1/2n}\\
&\quad=n!^{-1/2n}\,\Gamma_n(S;\mu). \quad\blacksquare
\end{aligned}$$

In the case $S=I_X$ we will use a "two sided version" of Lemma 2.1. For this purpose we define the covariance operator $T_\mu\in\mathscr{L}(X, X')$ for a measure $\mu\in W(B_{X'})$ by

$$\langle x, T_\mu y\rangle := \int_{B_{X'}}\langle x, a\rangle\langle y, a\rangle\, d\mu(a).$$

LEMMA 2.2. *Let $\mu\in W(B_{X'})$. Then*

$$\Gamma_n(T_\mu)=n!^{-1/n}\,\Gamma_n(X;\mu)^2.$$

Proof. By local reflexivity and again by Lemma 1.3 we derive

$$\begin{aligned}
&\Gamma_n(T_\mu)\\
&\quad=\sup\{|\det(\langle x_i, T_\mu y_j\rangle)_{i,j=1}^n|^{1/n}: x_i, y_j\in B_X\}\\
&\quad=\sup\{|(x_1\wedge\cdots\wedge x_n, \bar y_1\wedge\cdots\wedge\bar y_n)_{\Lambda_2^n L_2(B_{X'};\mu)}|^{1/n}: x_i, y_j\in B_X\}\\
&\quad=\sup\{|(x_1\wedge\cdots\wedge x_n, \bar x_1\wedge\cdots\wedge\bar x_n)_{\Lambda_2^n L_2(B_{X'},\mu)}|^{1/n}: x_i\in B_X\}\\
&\quad=\sup\left\{\frac{1}{n!}\int_{B_{X'}}\cdots\int_{B_{X'}}|\det(\langle x_i, a_j\rangle)_{i,j=1}^n|^2 d\mu(a_1)\cdots d\mu(a_n): x_i\in B_X\right\}^{1/n}\\
&\quad=n!^{-1/n}\,\Gamma_n(X;\mu)^2. \quad\blacksquare
\end{aligned}$$

Weaker, but more general, variants of Lemmas 2.1 and 2.2 are also useful. Moreover, they improve [2, (2.1, 2.5)].

LEMMA 2.1′. *Let $S\in\mathscr{L}(X, Y)$ and let $T\in\Pi_2(Y, Z)$ be dominated by $\mu\in W(B_{Y'})$. Then*

$$\Gamma_n(TS)\leqslant n!^{-1/2n}\,\Gamma_n(S;\mu)\,\pi_2(T).$$

Proof. Applying Theorem 1.1 to $s = Sx_1 \wedge \cdots \wedge Sx_n$ and taking the supremum over $x_i \in B_X$ we arrive at our assertion. ∎

LEMMA 2.2′. *Let $A \in \mathscr{L}(X_0, X)$, $S \in \Gamma_2^*(X, Y)$, and $B \in \mathscr{L}(Y, Y_0)$. If S is dominated by $\mu \in W(B_{X'})$ and $v \in W(B_{Y''})$, then*

$$\Gamma_n(BSA) \leqslant n!^{-1/n}\, \Gamma_n(B'; v)\, \Gamma_n(A; \mu)\, \gamma_2^*(S).$$

Proof. We assume $S = S_2 S_1$ with $S_1 \in \Pi_2(X, H)$ and $S_2' \in \Pi_2(Y', H')$ such that $\gamma_2^*(S) = \pi_2(S_1)\, \pi_2(S_2')$ (μ and v dominate S_1 and S_2, respectively). Setting

$$s = Ax_1^0 \wedge \cdots \wedge Ax_n^0 \qquad \text{and} \qquad t = B'b_1^0 \wedge \cdots \wedge B'b_n^0$$

we obtain

$$\begin{aligned}
&|\det(\langle BSAx_i^0, b_j^0 \rangle)_{i,j=1}^n| \\
&\quad = |\det(\langle S_1 Ax_i^0, S_2' B' b_j^0 \rangle)| \\
&\quad = |(S_1 Ax_1^0 \wedge \cdots \wedge S_1 Ax_n^0, S_2' B' b_1^0 \wedge \cdots \wedge S_2' B' b_n^0)_{A_2^n H}| \\
&\quad \leqslant \tau(S_1 Ax_1^0 \wedge \cdots \wedge S_1 Ax_n^0)\, \tau(S_2' B' b_1^0 \wedge \cdots \wedge S_2' B' b_n^0) \\
&\quad \leqslant n!^{-1}\, \pi_2(S_1)^n\, \pi_2(S_2')^n \\
&\qquad \times \left(\int_{B_{X'}} \cdots \int_{B_{X'}} |\det(\langle Ax_i^0, a_j \rangle)|^2\, d\mu(a_1) \cdots d\mu(a_n) \right)^{1/2} \\
&\qquad \times \left(\int_{B_{Y''}} \cdots \int_{B_{Y''}} |\det(\langle B'b_j^0, y_k'' \rangle)|^2\, dv(y_1'') \cdots dv(y_n'') \right)^{1/2}
\end{aligned}$$

from Theorem 1.2. Passing to the supremum over $x_i^0 \in B_{X_0}$ and $b_j^0 \in B_{Y_0'}$ yields the desired result. ∎

Before we consider some examples we derive two basic properties of the modified Grothendieck numbers which are needed in the sequel.

COROLLARY 2.3. *Let $S \in \mathscr{L}(X, Y)$ and $\mu \in W(B_{Y'})$. Then*

$$\Gamma_n(S; \mu) \leqslant n!^{1/2n}\, \|S\|.$$

Proof. Using Lemma 2.1 and [2] we obtain

$$\Gamma_n(S; \mu) = n!^{1/2n}\, \Gamma_n(JS) \leqslant n!^{1/2n}\, \|JS\| \leqslant n!^{1/2n}\, \|S\|. \quad ∎$$

COROLLARY 2.4. *Let $Y \subseteq X$ be Banach spaces and let $v \in W(B_{Y'})$. Then there exists a measure $\mu \in W(B_{X'})$ such that*

$$\Gamma_n(Y; v) \leqslant \Gamma_n(X; \mu) \qquad \textit{for} \quad n = 1, 2, \ldots.$$

Proof. If I: $Y \to X$ and J: $Y \to L_2(B_{Y'}; v)$ are the canonical embeddings and if $\tilde{J}$: $X \to L_2(B_{Y'}; v)$ is an extension of J with $\pi_2(\tilde{J}) = \pi_2(J) = 1$ and the dominating measure $\mu \in W(B_{X'})$, then

$$\Gamma_n(Y; v) = n!^{1/2n}\, \Gamma_n(J) = n!^{1/2n}\, \Gamma_n(\tilde{J}I)$$
$$\leqslant \Gamma_n(I; \mu)\, \pi_2(\tilde{J}) \leqslant \Gamma_n(X; \mu)$$

according to Lemmas 2.1 and 2.1′. ∎

Now we are in a position to treat some examples. For the first one we mention $\Gamma_n(l_2^n) = 1$ according to [2].

EXAMPLE 2.5. *Let $\mu \in W(B_{l_2^n})$* and $\{e_i\}$ be the standard basis of l_2^n. Then

$$\left(\int_{B_{l_2^n}} \cdots \int_{B_{l_2^n}} |\det(\langle e_i, a_j \rangle)_{i,j=1}^n|^2\, d\mu(a_1) \cdots d\mu(a_n) \right)^{1/2n}$$
$$= \Gamma_n(l_2^n; \mu) \leqslant \left(\frac{n!}{n^n} \right)^{1/2n}.$$

In the case in which μ is the Haar measure on the sphere S_{n-1} or $\mu = 1/n \sum_{j=1}^n \delta_{e_j}$ equality holds.

Proof. By the volume and multiplication properties of the determinant it is easy to see that

$$\sup\{|\det(\langle x_i, a_j \rangle)| : x_i \in B_{l_2^n}\} = |\det(\langle e_i, a_j \rangle)|$$

such that

$$\left(\int_{B_{l_2^n}} \cdots \int_{B_{l_2^n}} |\det(\langle e_i, a_j \rangle)_{i,j=1}^n|^2\, d\mu(a_1) \cdots d\mu(a_n) \right)^{1/2n} = \Gamma_n(l_2^n; \mu).$$

On the other hand, by Lemma 2.1 and [2] we obtain

$$\Gamma_n(l_2^n; \mu) = n!^{1/2n}\, \Gamma_n(J: l_2^n \to L_2(B_{l_2^n}; \mu))$$
$$= n!^{1/2n}\, (a_1(J) \cdots a_n(J))^{1/n},$$

where $a_k(J)$ are the usual approximation numbers of J (see Section 3). With the help of [11, (2.11.24)] we continue to

$$\Gamma_n(l_2^n;\mu)\leqslant\left(\frac{n!}{n^n}\right)^{1/2n}(a_1(J)^2+\cdots+a_n(J)^2)^{1/2}\leqslant\left(\frac{n!}{n^n}\right)^{1/2n}\pi_2(J)$$

$$\leqslant\left(\frac{n!}{n^n}\right)^{1/2n}.$$

Now let μ be the Haar measure on S_{n-1} or $\mu=1/n\sum_{j=1}^n\delta_{e_j}$. In both cases the covariance operator $T_\mu\colon l_2^n\to l_2^n$ satisfies

$$\langle e_i, T_\mu e_j\rangle=\int_{B_{l_2^n}}\alpha_i\alpha_j\,d\mu(\{\alpha_1,...,\alpha_n\})=1/n\,\delta_{ij}.$$

Hence $T_\mu=1/nI$. Applying Lemma 2.2 yields

$$\Gamma_n(l_2^n;\mu)=(n!)^{1/2n}\,\Gamma_n(1/nI)^{1/2}=n^{-1/2}(n!)^{1/2n}. \quad \blacksquare$$

For later use we construct measures $\mu\in W(B_{l_2^n})$ with

$$\int_{B_{l_2^n}}\cdots\int_{B_{l_2^n}}|\det(\langle e_i, a_j\rangle)_{i,j=1}^n|^2\,d\mu(a_1)\cdots d\mu(a_n)=\frac{n!}{n^n}$$

in a more general way using ellipsoids of maximal volume.

Let E be an n-dimensional Banach space. We will say that $u\in\mathscr{L}(l_2^n, E)$ is a *John-map*, if $\|u\|=1$ and $\pi_2(u^{-1})=n^{1/2}$. Note that the image $u(B_{l_2^n})$ is the unique ellipsoid of maximal volume which is contained in B_E.

EXAMPLE 2.6. Let E be an n-dimensional Banach space and let $u\in\mathscr{L}(l_2^n, E)$ be a John-map. Furthermore, let u^{-1} be dominated by $\mu\in W(B_{E'})$ and let $\nu\in W(B_{l_2^n})$ be the image measure of μ with respect to $u'\in\mathscr{L}(E', l_2^n)$. Then

$$\left(\int_{B_{l_2^n}}\cdots\int_{B_{l_2^n}}|\det(\langle e_i, a_j\rangle)_{i,j=1}^n|^2\,d\nu(a_1)\cdots d\nu(a_n)\right)^{1/2n}$$

$$=\Gamma_n(u;\mu)=\left(\frac{n!}{n^n}\right)^{1/2n}.$$

Proof. The left-hand equality follows from

$$\int_{B_{l_2^n}} \cdots \int_{B_{l_2^n}} |\det(\langle e_i, a_j\rangle)_{i,j=1}^n|^2 \, dv(a_1) \cdots dv(a_n)$$

$$= \int_{B_{E'}} \cdots \int_{B_{E'}} |\det(\langle e_i, u'b_j\rangle)_{i,j=1}^n|^2 \, d\mu(b_1) \cdots d\mu(b_n)$$

$$= \sup \left\{ \int_{B_{E'}} \cdots \int_{B_{E'}} |\det(\langle x_i, u'b_j\rangle)_{i,j=1}^n|^2 \, d\mu(b_1) \cdots d\mu(b_n) \colon x_i \in B_{l_2^n} \right\},$$

using the same argument as that given in the proof of Example 2.5. We consider the right-hand equality. From the construction of the John-map it is clear that $J\colon E \to L_2(B_{E'}; \mu)$ considered as a map on the image $J(E)$ and $n^{-1/2}u^{-1}$ may be identified. Hence

$$\Gamma_n(u; \mu) = n!^{1/2n} \, \Gamma_n(n^{-1/2}u^{-1}u) = \left(\frac{n!}{n^n}\right)^{1/2n} \Gamma_n(l_2^n) = \left(\frac{n!}{n^n}\right)^{1/2n}$$

according to Lemma 2.1. ∎

Another example we want to discuss is

EXAMPLE 2.7. Let $\mu \in W(B_{l_\infty})$ and let $\{e_i\}$ be the standard basis of l_1. Then

$$\sup_{i_1 < \cdots < i_n} \left\{ \int_{B_{l_\infty}} \cdots \int_{B_{l_\infty}} |\det(\langle e_{i_k}, a_l\rangle)_{k,l=1}^n|^2 \, d\mu(a_1) \cdots d\mu(a_n) \right\}^{1/2n}$$

$$= \Gamma_n(l_1; \mu) \leqslant n!^{1/2n}.$$

If μ is induced by the embedding $J\colon [\{-1, +1\}^{\mathbb{N}}, v] \to B_{l_\infty}$, where v is the normalized Haar measure on the product group $\{-1, +1\}^{\mathbb{N}}$, and if ε_{ij} is a family of independent random variables on $[\Omega, \mathscr{F}, P]$ with $P(\varepsilon_{ij} = 1) = P(\varepsilon_{ij} = -1) = \frac{1}{2}$ then

$$\left(\int_\Omega |\det(\varepsilon_{ij})_{i,j=1}^n|^2 \, dP(\omega) \right)^{1/2n} = \Gamma_n(l_1; v) = n!^{1/2n}.$$

Proof. Let $\mu \in W(B_{l_\infty})$ be arbitrary. Defining $t\colon l_1 \times \cdots \times l_1 \to \mathbb{R}$ by

$$t(x_1, ..., x_n) := \left(\int_{B_{l_\infty}} \cdots \int_{B_{l_\infty}} |\det(\langle x_i, a_j\rangle)|^2 \, d\mu(a_1) \cdots d\mu(a_n) \right)^{1/2}$$

we obtain a map which is continuous and convex in each component. Therefore

$$\Gamma_n(l_1;\mu)=\sup\{t(e_{i_1},...,e_{i_n})^{1/n}: i_1<\cdots<i_n\}.$$

The estimate $\Gamma_n(l_1;\mu)\leqslant n!^{1/2n}$ follows from Corollary 2.3. Now we assume μ to be the image of the Haar measure ν on $\{-1,+1\}^{\mathbb{N}}$. The continuity of J and the regularity of ν imply the regularity of μ. The symmetry of μ yields $\Gamma_n(l_1;\mu)=t(e_1,...,e_n)^{1/n}$. Hence

$$\Gamma_n(l_1;\mu)=\left(\int_{(-1,1)^N}\cdots\int_{(-1,1)^N}|\det(\langle e_i, Jb_j\rangle)|^2\,d\nu(b_1)\cdots d\nu(b_n)\right)^{1/2n}$$
$$=\left(\int_\Omega |\det(\varepsilon_{ij})_{i,j=1}^n|^2\,dP(\omega)\right)^{1/2n}.$$

To compute $\Gamma_n(l_1;\mu)$ we consider the covariance operator $T_\mu: l_1\to l_\infty$. It is not hard to check that

$$\int_{B_{l_\infty}}\langle e_i, a\rangle\langle e_j, a\rangle\,d\mu(a)=\delta_{ij}.$$

Consequently $T_\mu=I: l_1\to l_\infty$ such that

$$\Gamma_n(l_1;\mu)=n!^{1/2n}\,\Gamma_n(I)^{1/2}=n!^{1/2n}$$

according to Lemma 2.2 and

$$\Gamma_n(I: l_1\to l_\infty)=\sup\{|\det(\langle Ie_{i_k}, e_{j_l}\rangle)_{k,l=1}^n|^{1/n}: i_k, j_l\in\mathbb{N}\}=1$$

(again use convexity and continuity). ∎

Corollaries 2.3, 2.4 and Example 2.7 yield at once

COROLLARY 2.8. *Let X be a Banach space which contains l_1 isometrically. Then there exists a measure $\mu\in W(B_{X'})$ such that*

$$\Gamma_n(X;\mu)=n!^{1/2n}\qquad \textit{for}\quad n=1,2,...$$

In the next section we see that the above property is typical for Banach spaces containing an isomorphic copy of l_1.

3. Relations to the Geometry of Banach Spaces

We will show that the asymptotic behaviour of the modified Grothendieck numbers $\Gamma_n(X;\mu)$ characterizes some classes of Banach spaces. As a basic tool we make use of the approximation numbers, which are defined as

$$a_n(S) := \inf\{\|S-L\|: L \in \mathscr{L}(X, Y), \operatorname{rank}(L) < n\}$$

for an operator $S \in \mathscr{L}(X, Y)$. In the following it is convenient to set

$$\mathscr{L}^a_{p,q} := \{S \in \mathscr{L}(X, Y): \{n^{1/p-1/q} a_n(S)\} \in l_q\}$$

for $0 < p < \infty$ and $0 < q \leqslant \infty$.

With the help of the following lemma we will translate known results about approximation numbers of absolutely 2-summing operators into the language of Grothendieck numbers.

LEMMA 3.1. *Let $S \in \mathscr{L}(X, Y)$, $\mu \in W(B_{Y'})$, and J: $Y \to L_2(B_{Y'}; \mu)$ be the canonical embedding. Then*

$$a_1(JS) \cdots a_n(JS) \leqslant n!^{-1/2}\, \Gamma_n(S; \mu)^n \leqslant c^n \dot{a}_1(JS) \cdots \dot{a}_n(JS),$$

where $c > 0$ is an absolute constant and $\{\dot{a}_k(JS)\}$ stands for the doubled sequence $\{a_1(JS), a_1(JS), a_2(JS), a_2(JS), ...\}$.

Proof. Since

$$a_1(JS) \cdots a_n(JS) \leqslant \Gamma_n(JS)^n \leqslant c^n \dot{a}_1(JS) \cdots \dot{a}_n(JS)$$

according to [3, (2.2)] our assertion follows from Lemma 2.1. ∎

The left-hand side of Lemma 3.1 can be formulated more generally.

LEMMA 3.1′. *Let $S \in \mathscr{L}(X, Y)$ and let $T \in \Pi_2(Y, Z)$ be dominated by $\mu \in W(B_{Y'})$. Then for all $n = 1, 2, ...$*

$$(a_1(TS) \cdots a_n(TS))^{1/n} \leqslant n!^{-1/2n}\, \Gamma_n(S; \mu)\, \pi_2(T).$$

Proof. Considering the factorization $T = BJ$, where J: $Y \to L_2(B_{Y'}; \mu)$ is as usual and where $\pi_2(T) = \|B\|$, we obtain

$$(a_1(TS) \cdots a_n(TS))^{1/n} \leqslant (a_1(JS) \cdots a_n(JS))^{1/n} \|B\|$$
$$\leqslant n!^{-1/2n}\, \Gamma_n(S; \mu)\, \pi_2(T)$$

from Lemma 3.1. ∎

Let $0 \leqslant \alpha \leqslant \frac{1}{2}$. Then all Banach spaces X such that

$$\sup_n n^{-\alpha} \Gamma_n(X) < \infty$$

form a well-known class of Banach spaces. For $\alpha=0$ we obtain the weak Hilbert spaces; $\alpha=\frac{1}{2}$ yields the class of all Banach spaces. An L_p-space belongs to the above class whenever $\alpha=|1/p-\frac{1}{2}|$ (see [2, 3, 7, 12, 15]).

With respect to the above classes the different Grothendieck numbers possess the same behaviour.

THEOREM 3.2. *Let X be a Banach space and $0\leqslant\alpha\leqslant\frac{1}{2}$. Then $\sup_n n^{-\alpha}\Gamma_n(X)<\infty$ if and only if*

$$\sup_n n^{-\alpha}\Gamma_n(X;\mu)<\infty \qquad \textit{for all} \quad \mu\in W(B_{X'}).$$

Proof. Since $\Gamma_n(X;\mu)\leqslant\Gamma_n(X)$ we show one direction only. If Y is an arbitrary Banach space and $S\in\Pi_2(X,Y)$, then we obtain $\{a_n(S)\}_{n=1}^{\infty}\in l_{p,\infty}$ for $1/p=\frac{1}{2}-\alpha$ from Lemma 3.1′. Hence $\sup_n n^{-\alpha}\Gamma_n(X)<\infty$ according to [7, (4.5)] or [14, (2.2)]. ∎

The asymptotic behaviours of $\Gamma_n(X;\mu)$ and $\Gamma_n(X)$ are not always the same. For example, $\Gamma_n(X)\geqslant 1$ whenever $\dim(X)\geqslant n$ or in [13] it is shown that

$$\Gamma_n(X)\geqslant cn^{1/2} \qquad \text{if and only if} \quad X \text{ is not } K\text{-convex.}$$

In contrast to this we have the following two results.

THEOREM 3.3. *A Banach space X is isomorphic to a Hilbert space if and only if*

$$\sum_n \Gamma_n(X;\mu)^2/n<\infty \qquad \textit{for all} \quad \mu\in W(B_{X'})$$

and (**)

$$\sum_n \Gamma_n(X';\nu)^2/n<\infty \qquad \textit{for all} \quad \nu\in W(B_{X''}).$$

Proof. From Lemma 3.1 ($S=I_X$) and from the factorization argument given in the proof of Theorem 3.2 it is clear that (*) is equivalent to

$$\Pi_2(X,Y)\subseteq\mathscr{L}^a_{2,2}(X,Y) \qquad \text{and} \qquad \Pi_2(X',Y)\subseteq\mathscr{L}^a_{2,2}(X',Y) \qquad (**)$$

for all Banach spaces Y. Hence (*) is fulfilled whenever X is isomorphic to a Hilbert space. Let us treat the converse. The second inclusion of (**) implies $\{a_n(S)\}\in l_2$ for all $S\colon Z\to X$ with $S'\in\Pi_2$. Hence the definition of Γ_2^* and the multiplicity of the approximation numbers imply $N(X,X)\subseteq\Gamma_2^*(X,X)\subseteq\mathscr{L}^a_{1,1}(X,X)$. Therefore X is a Hilbert space according to [5, Theorem 3.15]. ∎

Problem. Does "$\Pi_2(X, Y) \subseteq \mathscr{L}^a_{2,2}(X, Y)$ for all Banach spaces Y" imply that X is a Hilbert space? From Theorem 3.2 we know that X must be a weak Hilbert space.

THEOREM 3.4. *For a Banach space X the following are equivalent.*

(1) *X contains an isomorphic copy of l_1.*

(2) *There exist $\mu \in W(B_{X'})$ and $c > 0$ such that*

$$\Gamma_n(X; \mu) \geqslant cn^{1/2} \qquad \text{for} \quad n = 1, 2, \ldots.$$

(3) *There exist $\mu \in W(B_{X'})$ and $\alpha, \beta > 0$ such that for all $n = 1, 2, \ldots$ there are $x_1, \ldots, x_n \in B_X$ with*

$$\mu \times \cdots \times \mu\{(a_1, \ldots, a_n): |\det(\langle x_i, a_j \rangle)|^{1/n} \geqslant \alpha n^{1/2}\} \geqslant \beta^n.$$

Proof. (1) $\Leftrightarrow$ (2). A result of Pełczynski and Ovsepian [8, Proposition 3] says that a Banach space X contains l_1 if and only if there exists a non-compact operator S: $X \to l_2 \in \Pi_2$. Hence Lemma 3.1 yields the equivalence. ((1) $\Rightarrow$ (2) follows directly from Example 2.7 and Corollary 2.4 in a more constructive way.)

(2) $\Rightarrow$ (3). We choose $x_1, \ldots, x_n \in B_X$ with

$$\int_{B_{X'}} \cdots \int_{B_{X'}} |\det(\langle x_i, a_j \rangle)|^2 \, d\mu(a_1) \cdots d\mu(a_n) \geqslant (c/2)^{2n} n^n.$$

Defining $\alpha := c/4$ and

$$p := \mu \times \cdots \times \mu\{(a_1, \ldots, a_n): |\det(\langle x_i, a_j \rangle)|^{1/n} \geqslant \alpha n^{1/2}\}$$

we conclude

$$(c/2)^{2n} n^n \leqslant (1-p)\, \alpha^{2n} n^n + p n^n \leqslant ((c/4)^{2n} + p) n^n.$$

Hence $p \geqslant (c/2)^{2n} - (c/4)^{2n} \geqslant (c/4)^{2n}$ and $\beta := (c/4)^2$ satisfies (3).

(3) $\Rightarrow$ (2). This is clear since $\Gamma_n(X; \mu)^{2n} \geqslant \alpha^{2n} \beta^n n^n$. ∎

It is known that an operator $S \in \mathscr{L}(X, Y)$ is compact if and only if the sequence of its Gelfand numbers

$$c_n(S) = \inf\{\|S|_E\| : E \subseteq X, \operatorname{codim}(E) < n\}$$

tends to zero. The same holds for the Kolmogorov numbers

$$d_n(S) := \inf\{\|Q_F S\| : F \subseteq Y, \dim(F) < n, Q_F: Y \to Y/F \text{ canonical}\}.$$

Now the result of Ovsepian and Pełczynski [8] can be formulated as follows.

A Banach space X does not contain l_1 if and only if $c_n(S) \xrightarrow[n]{} 0$ ($d_n(S) \xrightarrow[n]{} 0$) for all $S \in \Pi_2(X, Y)$ and all Banach spaces Y. Moreover, it is clear that $c_n(S) \xrightarrow[n]{} 0$ (or $d_n(S) \xrightarrow[n]{} 0$) for all $S \in \Gamma_2^*(X, Y)$ if X or Y' does not contain l_1.

We will replace the Gelfand (or Kolmogorov) numbers by the Grothendieck numbers. In general we have

$$(c_1(S) \cdots c_n(S))^{1/n} \leqslant \Gamma_n(S) \qquad \text{and} \qquad (d_1(S) \cdots d_n(S))^{1/n} \leqslant \Gamma_n(S)$$

for all $S \in \mathscr{L}(X, Y)$ and all Banach spaces X, Y (this is a result of Carl; cf. [3]). The converse does not hold in this form since, for example,

$$c_n(I\colon l_1^m \to l_\infty^m) \leqslant 6\,\frac{m^{1/2}}{n} \qquad \text{and} \qquad d_n(I\colon l_1^m \to l_\infty^m) \leqslant 6\,\frac{m^{1/2}}{n}$$

for $n = 1, ..., m$ (cf. [10, (11.11.11)]) whereas $\Gamma_n(I\colon l_1^m \to l_\infty^m) = 1$.

THEOREM 3.5. *For a Banach space X the following are equivalent.*

(1) *X does not contain an isomorphic copy of l_1.*

(2) *For all Banach spaces Y and for all $S \in \Pi_2(X, Y)$ we have*

$$\Gamma_n(S) \xrightarrow[n]{} 0.$$

(3) *For all Banach spaces Y, for all $S \in \Pi_2(X, Y)$, and for all sequences $\{x_n\} \subseteq B_X$ we have*

$$(\varepsilon(Sx_1 \wedge \cdots \wedge Sx_n))^{1/n} \xrightarrow[n]{} 0.$$

Proof. (1) $\Rightarrow$ (2). If X does not contain l_1 then $n^{-1/2}\Gamma_n(X; \mu) \xrightarrow[n]{} 0$ for all $\mu \in W(B_{X'})$ according to Lemma 3.1 and [8]. Hence (2) follows from Lemma 2.1′.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (1). We assume that X contains a copy of l_1, say $Y \subseteq X$. If $\{y_n\}$ corresponds to the standard basis of l_1, the operator $S\colon Y \to l_2$ defined by $Sy_i := e_i$ is absolutely 2-summing (cf. [10, (22.4.4)]). It is known that there exists an extension $T\colon X \to l_2 \in \Pi_2$. Hence

$$\varepsilon(Ty_1 \wedge \cdots \wedge Ty_n) = \varepsilon(e_1 \wedge \cdots \wedge e_n) = 1 \qquad \text{for all} \quad n = 1, 2, ...,$$

which is a contradiction to (3). ∎

Furthermore, from Lemmas 2.2′, 3.1 and [8] we obtain

THEOREM 3.6. *Let X and Y be a Banach spaces such that at least one of the spaces X and Y' does not contain an isomorphic copy of l_1. Then*

$$\Gamma_n(S) \xrightarrow[n]{} 0 \qquad \textit{for all} \quad S \in \Gamma_2^*(X, Y).$$

Remark. The converse of Theorem 3.6 is not true. If we set $X = Y = l_1$ all operators $S \in \Gamma_2^*(X, Y)$ factor as $S = BA$ with $A \in \mathscr{L}(l_1, l_2)$ and $B \in \mathscr{L}(l_2, l_1)$. B is known to be automatically compact (cf. [6, (I.2.c.3)]) such that $\Gamma_n(B) \xrightarrow[n]{} 0$ according to [3, (2.2)]. Hence $\Gamma_n(S) \leqslant \Gamma_n(B) \|A\|$ implies $\Gamma_n(S) \xrightarrow[n]{} 0$.

4. CUBICAL VOLUME RATIO

We demonstrate that the Grothendieck numbers are useful for considering the cubical volume ratio of convex and symmetric bodies in $\mathbb{R}^n$. We reprove a result of Pełczynski and Szarek [9, Corollary 2.2] and use the estimates, obtained for this purpose, to sharpen the relation between the Grothendieck numbers and the volume ratio using ellipsoids of maximal and minimal volume.

As in [9] we also use in Proposition 4.2 the Gauss-inequality. Nevertheless our approach seems to be somewhat different and yields further consequences.

From now on all Banach spaces are assumed to be real. The volume of a body $C \subseteq E$, where E is a finite-dimensional Banach space, is taken with respect to a fixed non-trivial Haar-measure and denoted by $|C|$. For simplicity we take the standard Lebesgue measure in the case $E = l_2^n$ or $E = l_\infty^n$.

The *cubical volume ratio* of the unit ball B_E of an n-dimensional Banach space E is defined as

$$a(E) := \sup \left\{ \frac{|v(B_E)|}{|B_{l_\infty^n}|} : \|v: E \to l_\infty^n\| \leqslant 1 \right\}^{1/n}.$$

By an ellipsoid D in E we mean the image of $B_{l_2^n}$ under some $u \in \mathscr{L}(l_2^n, E)$, that is, $D = u(B_{l_2^n})$. $D_{\max}^E \subseteq B_E$ is the ellipsoid of maximal volume which lies in B_E and $D_{\min}^E \supseteq B_E$ the ellipsoid of minimal volume which contains B_E.

With the above notation we define the usual *volume ratio* of E as

$$\mathrm{vr}(E) := \left(\frac{|B_E|}{|D_{\max}^E|} \right)^{1/n}.$$

The following easy observation is the reason for the use of Grothendieck numbers to compare the cubical volume ratio with the usual volume ratio.

LEMMA 4.1. *Let E be n-dimensional. Then*

$$a(E) = a(l_2^n)\,\mathrm{vr}(E)\,\Gamma_n(u),$$

where $u \in \mathscr{L}(l_2^n, E)$ is a John-map ($\|u\| = 1$, $\pi_2(u^{-1}) = n^{1/2}$).

Proof. From the definition of $a(E)$ we obtain

$$a(E) = \sup\left\{\frac{|B_{l_2^n}|}{|B_{l_\infty^n}|}\,\frac{|B_E|}{|u(B_{l_2^n})|}\,\frac{|vu(B_{l_2^n})|}{|B_{l_2^n}|} : \|v: E \to l_\infty^n\| \leqslant 1\right\}^{1/n}$$

$$= a(l_2^n)\,\mathrm{vr}(E)\sup\left\{\frac{|vu(B_{l_2^n})|}{|B_{l_2^n}|} : \|v: E \to l_\infty^n\| \leqslant 1\right\}^{1/n}.$$

Using $|vu(B_{l_2^n})| = \Gamma_n(vu: l_2^n \to l_2^n)^n\,|B_{l_2^n}|$ and $\Gamma_n(vu: l_2^n \to l_2^n) = \Gamma_n(vu: l_2^n \to l_\infty^n)$ from [2] we continue to

$$a(E) = a(l_2^n)\,\mathrm{vr}(E)\sup\{\Gamma_n(vu): \|v: E \to l_\infty^n\| \leqslant 1\}$$

$$= a(l_2^n)\,\mathrm{vr}(E)\,\Gamma_n(u)$$

since $\Gamma_n(S) = \sup\{\Gamma_n(vS): \|v: Y \to l_\infty^n\| \leqslant 1\}$ for $S \in \mathscr{L}(X, Y)$ in general. ∎

Now we estimate $\Gamma_n(u)$ from below and from above. The estimate $\Gamma_n(u) \leqslant 1$ follows from [2] and is clearly the best possible.

PROPOSITION 4.2. *Let E be n-dimensional and let $u \in \mathscr{L}(l_2^n, E)$ be a John-map. Then for $N = n(n+1)/2$*

$$\left(\frac{N}{n}\right)^{1/2}\left(\frac{n!}{N(N-1)\cdots(N-n+1)}\right)^{1/2n} \leqslant \Gamma_n(u) \leqslant 1.$$

Proof. From [16, Theorem 15.5] we know that the inverse u^{-1} of a John-map can be dominated by a $\mu \in W(B_{E'})$ with $\mathrm{card}(\mathrm{supp}(\mu)) = N$. Setting $\mu = \sum_{j=1}^N \lambda_j\,\delta_{b_j}$ from Example 2.6 we obtain

$$\left(\frac{n!}{n^n}\right)^{1/2n} = \Gamma_n(u; \mu)$$

$$= \sup_{f_i \in B_{l_2^n}} \left\{ \int_{B_{E'}} \cdots \int_{B_{E'}} |\det(\langle uf_i, a_j \rangle)|^2 \, d\mu(a_1) \cdots d\mu(a_n) \right\}^{1/2n}$$

$$\leqslant \Gamma_n(u) \left(n! \sum_{j_1 < \cdots < j_n} \lambda_{j_1} \cdots \lambda_{j_n} \right)^{1/2n}.$$

We estimate the second factor from above by $((N \cdots (N-n+1))/N^n)^{1/2n}$ according to the Gauss-inequality [1, p. 11]. Hence the lower estimate of $\Gamma_n(u)$ follows. ▮

Directly from Lemma 4.1 and Proposition 4.2 we obtain

COROLLARY 4.3 [9, Corollary 2.2]. *Let E be n-dimensional. Then*

$$a(E) \leqslant a(l_2^n)\,\mathrm{vr}(E) \leqslant \left(\frac{n}{N}\right)^{1/2} \left(\frac{N \cdots (N-n+1)}{n!}\right)^{1/2n} a(E),$$

where $N = n(n+1)/2$.

We can also improve [3, Theorem 1.1].

COROLLARY 4.4. *Let X be a Banach space and let* $N = n(n+1)/2$. *Then*

$$\Gamma_n(X) \leqslant \sup \left\{ \frac{|D^E_{\min}|}{|D^E_{\max}|} \right\}^{1/n} \leqslant \frac{n}{N} \left(\frac{N \cdots (N-n+1)}{n!}\right)^{1/n} \Gamma_n(X),$$

where the supremum is taken over all $E \subseteq X$ *with* $\dim(E) = n$.

Proof. The proof is exactly the same as that in [3]. We have to replace the estimate $\Gamma_n(u) \geqslant 1/e$ by Proposition 4.2. ▮

ACKNOWLEDGMENTS

I thank Professors A. Pietsch and M. Junge for helpful hints during the preparation of this paper.

REFERENCES

1. E. BECKENBACH AND R. BELLMAN, "Inequalities," Springer-Verlag, Berlin/Heidelberg/New York, 1961.

2. S. GEISS, Grothendieck numbers of linear and continuous operators on Banach spaces, *Math. Nachr.* **148** (1990), 65–79.
3. S. GEISS, Grothendieck numbers and volume ratios of operators on Banach spaces, *Forum Math.* **2** (1990), 323–340.
4. J. R. HOLUB, Tensor product mappings, *Math. Ann.* **188** (1970), 1–12; *Proc. Amer. Math. Soc.* **42** (1974), 437–441.
5. W. B. JOHNSON, H. KONIG, B. MAUREY, AND J. R. RETHERFORD, Eigenvalues of p-summing and l_p-type operators in Banach spaces, *J. Funct. Anal.* **32** (1979), 353–380.
6. J. LINDENSTRAUSS AND L. TZAFRIRI, "Classical Banach Spaces I/II," Springer-Verlag, Berlin/Heidelberg/New York, 1977/1979.
7. V. MASCIONI, On weak cotype and weak type in Banach spaces, *Note Mat.* **8** (1988), 67–110.
8. R. I. OVSEPIAN AND A. PEŁCZYNSKI, The existence in every separable Banach space of a fundamental total and bounded biorthogonal sequence and related constructions of uniformly bounded orthonormal systems in L^2, *Sem. Maurey-Schwartz* (1973–1974), Exposé 20.
9. A. PEŁCZYNSI AND S. J. SZAREK, On parallelepipeds of minimal voume containing a convex symmetric body in $\mathbb{R}^n$, *Proc. Cambridge Philos. Soc.* **109** (1991), 125–148.
10. A. PIETSCH, "Operator Ideals," North-Holland, Amsterdam, 1980.
11. A. PIETSCH, "Eigenvalues and s-Numbers," Cambridge Univ. Press, Cambridge, 1987.
12. A. PIETSCH, Eigenvalue distributions and geometry of Banach spaces, *Math. Nachr.* **150** (1991), 41–81.
13. A. PIETSCH, Eigenvalue criteria for the B-convexity of Banach spaces, *J. London Math. Soc.* (2) **41**, No. 2 (1990), 310–322.
14. A. PIETSCH, Approximation numbers of nuclear operators and geometry of Banach spaces, *Arch. Math.* **57** (1991), 155–168.
15. G. PISIER, Weak Hilbert spaces, *Proc. London Math. Soc.* (3) **56** (1988), 547–579.
16. N. TOMCZAK-JAEGERMANN, "Banach–Mazur Distances and Finite Dimensional Operator Ideals," Pitman, London, 1988.